

What Does Conceptual Understanding Mean?

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Most of the people and professional organizations that advocate mathematics reform call for an increased emphasis on conceptual understanding. The MAA's new CUPM Curriculum Guide recommends:

- *All students, those for whom the (introductory mathematics) course is terminal and those for whom it serves as a springboard, need to learn to think effectively, quantitatively and logically.*
- *Students must learn with understanding, focusing on relatively few concepts but treating them in depth. Treating ideas in depth includes presenting each concept from multiple points of view and in progressively more sophisticated contexts.*
- *A study of these (disciplinary) reports and the textbooks and curricula of courses in other disciplines shows that the algorithmic skills that are the focus of computational college algebra courses are much less important than understanding the underlying concepts.*
- *Understanding as well as computational skills.* (Mathematical Association of America, 2004)

Similarly, the AMATYC Crossroads Standards state:

- *In general, emphasis on the meaning and use of mathematical ideas must increase, and attention to rote manipulation must decrease.*

2 What Does Conceptual Understanding Mean?

- *Faculty should include fewer topics but cover them in greater depth, with greater understanding, and with more flexibility. Such an approach will enable students to adapt to new situations.*
- *Areas that should receive increased attention include the conceptual understanding of mathematical ideas. (AMATYC, 1995)*

Several questions are asked about conceptual understanding:

- What does conceptual understanding mean, especially in introductory courses such as college algebra, precalculus or calculus?
- How is understanding recognized in student work?
- How does one develop a high level of conceptual understanding in students?
- How are courses altered to make conceptual understanding an important component?
- How should students be assessed to verify they have developed conceptual understanding?
- How do faculty reward students who display unexpected conceptual insights?

This article addresses each of these questions using a variety of specific examples and students' own words. Identifying whether students have achieved a particular level of algebraic skill is fairly easy—just give a variety of problems on an exam to determine whether or not students have mastered the desired skills. Determining whether students have achieved a level of conceptual understanding is considerably harder to assess.

A few years ago, one of the departments offered several sections of a traditional college algebra/trigonometry course along with several sections of a reform approach to college algebra/trigonometry. A number of common questions were placed on the final examination to compare how students from the two approaches would solve routine algebraic manipulation problems, non-routine conceptual problems and applied problems. One of the common problems reads:

Brookville College enrolled 2546 students in 1996 and 2702 students in 1998. Assume that enrollment follows a linear growth pattern.

- (a) *Write a linear equation that gives the enrollment in terms of the year t (let $t = 0$ represent 1996).*
- (b) *If the trend continues, what will the enrollment be in the year 2016?*
- (c) *What is the slope of the line you found in part (a)?*
- (d) *Explain, using an English sentence, the meaning of the slope here.*
- (e) *If the trend continues, when will the enrollment reach 3500 students?*

Both groups were equally adept at finding the slope and the equation of the line; however, several students in each group reversed the numerator and denominator in

the slope formula. The practical interpretation of slope in this context, as asked for in part (d), presents a very different image for what each group learned. In the reform classes, virtually every student was able to provide a meaningful interpretation of the slope in a well-constructed sentence. Several typical responses are listed:

- Every year, enrollment increases by 78 students
- The yearly increase of enrollment is 78 per year.
- The enrollment increases by 78 students every year.
- Every t year it will increase by that #.
- The slope is the growth in enrollment.

In comparison, the responses from the students in the traditional, skills-oriented classes were quite different. Only approximately one-third of these students were able to interpret the slope in a meaningful sentence. Another third ignored that part of the problem and did not respond. And, among the remaining responses, the following can be found:

- The difference in $(y_2 - y_1)/(x_2 - x_1)$
- Slope would be a constant increase or decreasing of a line. So if you would enroll 1 person a year the constant would be one and so would the slope.
- It is the change in students per year starting at 1996.
- Enrollments per year.
- The slope is the amount of the new students the school gets each year.
- The point in which the # of students is increasing.
- The meaning of the slope is the amount that is gained in years and students in a given amount of time.
- The ratio of students to the number of years.
- Difference of the y 's over the x 's.
- Since it is positive it increases.
- On a graph, for every point you move to the right on the x -axis. You move up 78 points on the y -axis.
- The slope in this equation means the students enrolled in 1996. $Y = MX + B$.
- The amount of students that enroll within a period of time.
- The change in the x -coordinates over the change in the y -coordinates.
- This is the rise in the number of students.
- The slope is the average amount of years it takes to get 156 more students enrolled in the school.
- Its how many times a year it increases.

Most students who had trouble interpreting the slope also had trouble using the equation of the line to answer the predictive questions. The complete set of student responses (for both groups) and a detailed analysis of those responses are discussed in Gordon (2001). A thorough analysis of all aspects of the comparison

4 What Does Conceptual Understanding Mean?

study, including a follow-up study of how the students from both groups performed in Calculus I, can be found in Gordon (2006).

Prior to administering the final examination, the instructors involved in teaching the traditional or reform sections were confident “their students” would do well with the examination. After reviewing the results, however, it was evident that, unless explicit attention was devoted to emphasizing the conceptual understanding of the slope (what it means) the majority of students were not able to create viable interpretations on their own. And, without that understanding, they were likely not able to apply the mathematics to realistic situations. Simply being able to perform the calculations to find the slope, or the equation of the line, clearly does not immediately translate into the ability to understand what the slope means in context, whether that context arise in one of their other courses in mathematics or courses in one of the quantitative disciplines or eventually on the job.

Moreover, if students are unable to make their own connections with a concept as simple as that of the slope of a line (which they have undoubtedly encountered in previous mathematics courses), it is unlikely that they will be able to create meaningful interpretations and connections on their own for more sophisticated mathematical concepts. For instance, what is the significance of the base (growth or decay factor) in an exponential function? What is the meaning of the power in a power function? What do the parameters in a realistic sinusoidal model tell about the phenomenon being modeled? What is the significance of the factors of a polynomial? What is the practical meaning of the derivative of a function? What is the significance of a definite integral?

Judging from the responses on slope, it is clear that we cannot simply concentrate on teaching the mathematical techniques that the students need. It is at least as important, if not more important, to stress conceptual understanding and the meaning of the mathematics. This can and should be accomplished by using realistic, contextual examples and problems and by forcing the students to think, not just to manipulate symbols. If we fail to do this, we are not adequately preparing them for successive mathematics courses, for courses in other disciplines, and for using mathematics on the job and throughout their lives.

“ x ” Marks the Spot

We are all familiar with the old cliché that “ x marks the spot.” Unfortunately, we have allowed mathematics education to reduce to something that is seemingly fixated only on that local spot while the uses of mathematics in all other disciplines focus globally on the entire universe of a through z , with the occasional contribution

of α through ω . Applications in other fields do not limit themselves to the use of x as the sole variable. Newton did not express his second law of motion as $y = mx$, nor did Einstein express the relationship between energy and mass as $y = c^2x$, and the ideal gas law does not say that $yz = nRx$. But mathematics students are trained to use x almost exclusively. In the few instances when other letters do appear, there is a fundamental disconnect in the minds of most students. They do not see how the mathematics they have learned has anything to do with this new situation. For instance, there is a good chance that they will recognize $4x^2 - 5x + 8 = 0$ as a quadratic equation, but $4p^2 - 5p + 8 = 0$ is, in their minds, something else entirely.

Some of this confusion is evident in calculus when the students are asked to set up optimization problems—they have difficulty with symbols other than x ’s. The confusion is much worse in courses in other fields where the variable x almost never surfaces. All too often, faculty in other departments complain that mathematics courses are too abstract. This is interpreted as too much emphasis on statements of theorems and their proofs or the use of delta-epsilon techniques. But the word “abstract” has a very different connotation outside mathematics; it refers to the fact that much of what is taught is done with no contexts or realistic applications and that all the work is done with the letter x , not letters that provide insight and cues into the situation.

For instance, Kepler’s third law expresses the relationship between the average distance of a planet from the sun and the length of its year. It can be written as $y^2 = 0.1664x^3$, in which case there are no suggestions of which variable represents which quantity—a student would have to either keep track of that information as he or she works through a problem or refer back to where the variables were originally defined (assuming that he or she bothered to do that). Alternatively, if Kepler’s law is written as $t^2 = 0.1664D^3$, which is the way it appears in physics classes, a huge conceptual hurdle for the students would be eliminated.

Unfortunately, this is not something that can be accomplished with just a handful of problems. If students see 50 exercises where the first 40 involve solving for x , the overriding impression is that x is the real variable and the few remaining cases are just there to torment them in some way.

Some Examples of Conceptual and Applied Problems

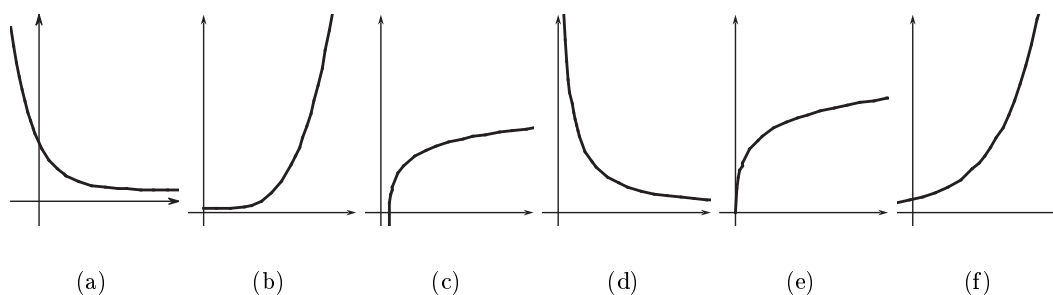
Our view is that conceptual understanding and realistic applications that make sense to the students are just two sides of the same coin—student understanding of the mathematics. Note that others believe that conceptual understanding does not necessarily involve applications and mathematical modeling. This is a manifestation

6 What Does Conceptual Understanding Mean?

of the healthy tension between pure and applied mathematics. What we do take exception to is the philosophy that introductory mathematics courses exist primarily to develop manipulative skills needed for subsequent courses. There is clearly a need for some manipulative skills. But, without understanding, those skills are useless. And, without some manipulative skills, a high degree of conceptual understanding is not particularly useful either.

In this section, a variety of conceptual problems, along with a discussion of each, illustrate the value to the students of asking each question.

1. Identify each of the following functions (a)–(n) as linear, exponential, power or something else. In each case, *explain* your reasoning.



(g) $y = 1.05^x$

(i) $P = (0.7)^t$

(k) $W = q^{(-1/2)}$

(h) $y = x^{1.05}$

(j) $z = w^{0.7}$

(l) $3U - 5V = 4$

(m)

x	y
0	3
1	5.1
2	7.2
3	9.3

(n)

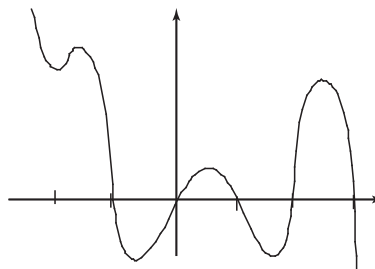
x	y
0	5
1	7
2	9.8
3	13.72

Discussion At first look, this problem likely appears trivial. But students actually find it very difficult. It goes to the heart of many of their problems in mathematics—the inability to recognize functional form, their difficulty with seeing beyond x and y , their inexperience with recognizing the behavioral patterns of different families of functions, etc. For instance, many students use the words “line” and “curve” interchangeably, so that any one of the functions in (a) through (f) can be declared “linear.” If they are using graphing

calculators, they must determine an appropriate window; if not, any smooth function appears linear on a small interval. So this type of problem actually keys on many of the major conceptual misunderstandings of students at this level.

2. For the polynomial shown,

- (a) What is the minimum degree? Give two different reasons for your answer.
- (b) What is the sign of the leading term? Explain.
- (c) What are the real roots?
- (d) What are the linear factors?
- (e) How many complex roots does the polynomial have?



Discussion Again, this conceptual problem appears to be very simple, but it requires students to know and understand the most important behavioral characteristics of polynomials—how the degree is related to the number of turning points and the number of inflection points, how to interpret the global behavior as it relates to the sign of the leading coefficient, the relationship between the real roots and the linear factors, and so forth.

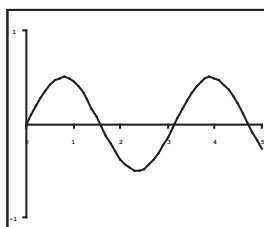
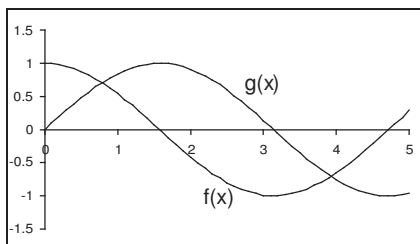
3. Two functions f and g are defined in the following table. Use the given values in the table to complete the table. If any entries are not defined, write “undefined.”

x	$f(x)$	$g(x)$	$f(x) - g(x)$	$f(x)/g(x)$	$f(g(x))$	$g(f(x))$
0	1	3				
1	0	1				
2	3	0				
3	2	2				

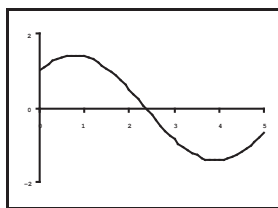
Discussion This is still another seemingly easy conceptual problem that students find quite difficult. It requires them to understand functional notation well and to understand the basic operations on functions, especially composite functions, very thoroughly. The lack of that understanding can be easily masked in a purely manipulative problem where a series of operations can be memorized and applied by rote without any understanding of what is happening.

8 What Does Conceptual Understanding Mean?

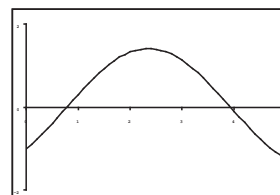
4. Two functions f and g are given in the accompanying figure. The following five graphs (a)–(e) are the graphs of $f + g$, $g - f$, $f \cdot g$, f/g , and g/f . Decide which is which and give reasons for your answers.



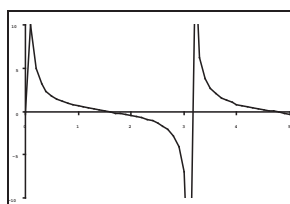
(a)



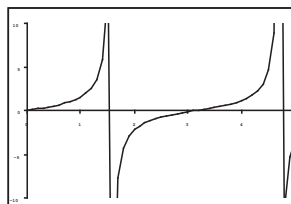
(b)



(c)



(d)



(e)

Discussion Once again, this conceptual problem requires that the students understand fully what operations on functions represents, not just how to move symbols around. The last two parts require a solid understanding of the properties of rational functions and how their behavior is dependent on the real roots of the numerator and the denominator.

5. The following table shows world-wide wind power generating capacity, in megawatts, in different years.

Year	1980	1985	1988	1990	1992	1995	1997	1999
Wind energy	10	1020	1580	1930	2510	4820	7640	13840

- Which variable is the independent variable and which is the dependent variable?
- Explain why an exponential function is the best model to use for this data.
- Find the exponential function that best fits this data.
- What are some reasonable values that you can use for the domain and range of this function?
- What is the practical significance of the base in the exponential function you created in part (c)?
- What is the doubling time for this exponential function? Explain what it means.
- According to your model, what do you predict for the total wind power generating capacity in 2010?

Discussion The notion of fitting functions to data using linear and nonlinear regression has become one of the hallmarks of reform courses in college algebra and precalculus. This problem indicates the substantial amount of deep mathematical understanding that comes to the fore in such activities; it is not just a matter of pushing a button on a calculator or using a spreadsheet function blindly to obtain a function. Issues about the behavioral characteristics of the basic families of functions are essential to knowing which function to choose. Issues regarding the choice of dependent and independent variables become fundamental—it is not just the simple x and y . Similarly, issues regarding the domain and range become central—it is no longer merely avoiding division by zero or the square root of a negative number, but a practical concern in each context about when the function makes sense and when it breaks down as a predictive tool. Further, students have to understand the meaning of the base of an exponential function as either a growth or a decay factor, as well as what the doubling time or the half life means. Finally, the predictive questions that naturally arise in such contexts lead to equations that are considerably more complicated than the standard exponential equations with one-digit positive integer constants for the base. For instance, in order to answer part (f), students have to solve the equation $52.497(1.373)^t = 2 \times 52.497$, where $t = 0$ in 1980.

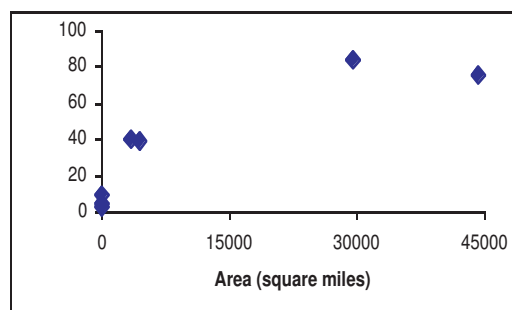
- Biologists have long observed that the larger the area of a region, the more species live there. The relationship is best modeled by a power function. Puerto Rico

10 What Does Conceptual Understanding Mean?

has 40 species of amphibians and reptiles on 3459 square miles and Hispaniola (Haiti and the Dominican Republic) has 84 species on 29,418 square miles.

- (a) Determine a power function that relates the number of species of reptiles and amphibians on a Caribbean island to its area.
 - (b) Use the relationship to predict the number of species of reptiles and amphibians on Cuba, which measures 44218 square miles.
7. Biologists have long observed the fact that the larger the area of a region, the more species that inhabit it. The accompanying table and associated scatterplot give some data on the area (in square miles) of various Caribbean islands in the Greater and Lesser Antilles and estimates on the number species of amphibians and reptiles living on each.

Island	Area	N
Redonda	1	3
Saba	4	5
Montserrat	40	9
Puerto Rico	3459	40
Jamaica	4411	39
Hispaniola	29418	84
Cuba	44218	76

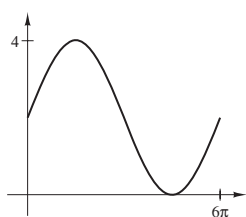


- (a) Which variable is the independent variable and which is the dependent variable?
- (b) The overall pattern in the data suggests either a power function with a positive power $p < 1$ or a logarithmic function, both of which are increasing and concave down. Explain why a power function is the better model to use for this data.
- (c) Find the power function that models the relationship between the number of species, N , living on one of these islands and the area, A , of the island and find the correlation coefficient.
- (d) What are some reasonable values that you can use for the domain and range of this function?
- (e) The area of Barbados is 166 square miles. Estimate the number of species of amphibians and reptiles living there.

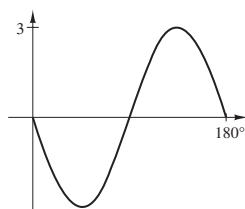
Discussion After linear and exponential functions, power functions are the most common type of function that arise in applications in most other disciplines. But the functions that arise are not as simple as $y = x^2$ or $y = x^3$. The

above two problems are much more closely attuned to the kind of problems that students will encounter in other quantitative courses than the more standard context-free problems one finds in traditional textbooks. As such, they provide better motivation to the students, as well as better preparation for other courses. At the same time, such problems require working with considerably more difficult equations that can challenge the students to utilize their algebraic ability rather than merely practice it.

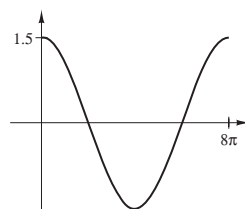
8. Write a possible formula for each of the following trigonometric functions:



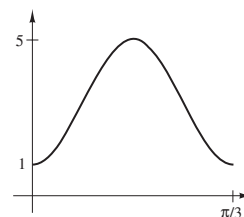
(a)



(b)



(c)



(d)

Discussion Traditional math courses typically ask students to draw the graph of relatively simple trigonometric functions of the form $y = 2 + 4\sin(3x)$. However, when students have graphing calculators at their fingertips, such problems become meaningless. On the other hand, the ability to construct a formula to match a pattern seen in a graph requires that students understand fully the specific role of each of the parameters in a general sinusoidal expression. At the same time, problems like this also build on the unifying theme throughout a course that one creates formulas for functions in order to answer predictive questions, so that the focus is always on creating functions, not producing graphs.

9. The average daytime high temperature in New York as a function of the day of the year varies between 32°F and 94°F . Assume the coldest day occurs on the 30th day and the hottest day on the 214th.

- (a) Sketch the graph of the temperature as a function of time over a three year time span.
- (b) Write a formula for a sinusoidal function that models the temperature over the course of a year.

12 What Does Conceptual Understanding Mean?

- (c) What are the domain and range for this function?
- (d) What are the amplitude, vertical shift, period, frequency, and phase shift of this function?
- (e) What is the most likely high temperature on March 15?
- (f) What are all the dates on which the high temperature is most likely 80° ?

Discussion The sine and cosine functions are used widely as the primary models for periodic processes. Contextual problems such as this one require a very deep understanding of the significance of each of the parameters in $y = A + B \sin(C(x - D))$. However, by working in a concrete context, each of the parameters has a well-defined meaning; they are not simply numbers associated with a bunch of words—amplitude, period, etc.—that students all too often confuse and use interchangeably.

10. You are buying a \$20,000 car and can choose between a \$2000 rebate and a 48 month loan at 4% or no rebate and a 36 month loan at 2%. Which should you choose? How did you make your decision? What would need to change in order for you to reverse your decision?

Discussion This problem represents a real-world situation in which people have to use mathematics to make an important personal decision. It does not merely ask the students to calculate a quantity, but is expressed in a more vague way, so that they have to identify what the situation is all about, decide on a procedure, then decide on the appropriate mathematical models to use, and finally interpret the answers.

11. Jack is setting up plans for a retirement fund. Write a difference equation for the balance b_n in this account for each of the following scenarios he is contemplating:
- (a) He will deposit \$3000 into an account guaranteed to pay 6% interest per year.
 - (b) He will deposit \$3000 initially into an account that pays 6% per year and then deposit an additional \$1000 each year.
 - (c) He will deposit \$3000 initially into the account and then increase his yearly contribution by \$1000 each year.
 - (d) He will deposit \$3000 initially into the account and then increase his yearly contribution by 10% each year.

Discussion This problem illustrates the idea of modeling a recursive process with a difference equation and the ease with which such models can be adapted

if one changes the underlying assumptions in the model. The use of such recursive thinking and the associated mathematical models based on difference equations has become a staple in most other fields, in large measure because of the universal availability of spreadsheets. The corresponding solutions of the difference equation can be generated either graphically or numerically using the difference equation capabilities of most graphing calculators or with a spreadsheet, to reinforce the concept that the solution of a difference equation is a function of n . At a later stage in such a course, students could be asked to construct the closed-form algebraic solution when appropriate solution techniques become available.

12. Suppose that your car has a 14 gallon gas tank that you fill as soon as the level drops to half-full. Also, every time you fill up, you add one quart ($\frac{1}{4}$ gallon) of an additive that mixes thoroughly with the gas and is then used up along with the gas.
 - (a) Write a difference equation that models the amount of the additive A_n in the tank from one fill-up to the next.
 - (b) Use the difference equation to calculate the amount of additive in the tank over the first 10 fill-ups.
 - (c) Sketch the graph of A_n as a function of n based on the values from part (b). What does the behavior suggest?
 - (d) Find the limiting value for the amount of the additive in the tank as n increases indefinitely.
 - (e) Find the closed form solution of the difference equation.
 - (f) How would the difference equation and the limiting value change if you fill up when the tank is 25% full instead of 50% full?
 - (g) How would the limiting value change if your gas tank holds 16 gallons instead of 14 and you fill up when the tank is half full?

Discussion This problem illustrates the idea of modeling an everyday process using a simple difference equation. The students have to create the difference equation and know that the solution of the difference equation is given by the sequence of successive values calculated based on an initial value; that is, the solution is a function of n . They also have to know the limiting behavior of the modified exponential decay function that arises as the closed-form solution.

14 What Does Conceptual Understanding Mean?

Developing Conceptual Understanding

When one examines the problem sets in virtually any calculus textbook from 20 or 30 years ago, a clear pattern emerges—most problem sets started with some 50 or 75 routine exercises and were followed by relatively few problems that focused on conceptual understanding of the fundamental calculus ideas and some applications drawn from other disciplines. The message sent to the students was very clear: the important thing was to know how to perform the mechanics. And, all too often, instructors reinforced this message by assigning only manipulative problems for a variety of reasons: many students had trouble with them, so they needed more and more practice and reinforcement; such problems were easy to create and mark on tests; there never was enough time to dwell on the “nice” problems at the end of the problem sets. In comparison, the so-called “reform” calculus texts moved the conceptual and realistic application problems front and center to show that they should be the primary focus.

The same emphasis is needed in all courses, especially the introductory courses below calculus. It is not enough to add some emphasis on conceptual understanding at the end of the section in the textbook or to the last few problems in a problem set. Rather, conceptual understanding is something that should permeate all aspects of a course—it should be emphasized in all sets of in-class examples, on all homework assignments, on all project assignments, and most importantly, on all tests. Students must see that understanding the mathematics is important in the sense that their grades will depend on it. Otherwise, they will not view it as important, particularly since most students have been through prior mathematics courses where understanding was not emphasized or valued.

Recognizing and Rewarding Conceptual Understanding

From an instructor’s point of view, one of the most satisfying aspects of stressing conceptual understanding is the variety of totally unexpected questions and insights that students come up with. For instance, here are a few such items we have encountered:

- After leading one college algebra class to compile a list of behavioral characteristics of cubic polynomials, one student raised the question: “Is it true that every cubic is centered at its point of inflection?” When asked what she meant by that (to draw her out for the sake of the other students), her eyes screwed up as if she was trying to visualize the image in her mind and, with her hands moving in opposing directions, she responded “Well, if you start at the point of

inflection and move in both directions, don’t you trace out the identical path?” Sure you do! The fact that most professional mathematicians are not aware of this delightful fact only increases the significance of her insight. What’s even more amazing is that she came into the college algebra course with a particularly low self-assessment of her mathematical abilities; all her previous math teachers had convinced her that she had no aptitude for mathematics!

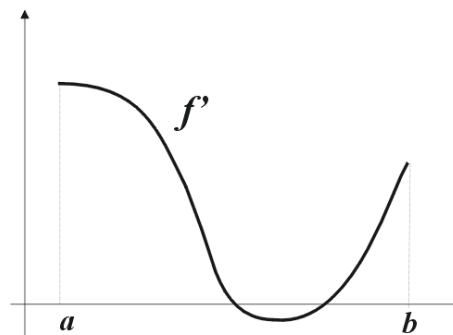
- One of the individual projects assigned in precalculus is based on a set of temperature measurements for Dallas taken every two weeks over the course of a year. The students have to construct a sinusoidal function that models these data. They usually come up with a variety of schemes for doing this. A typical formula looks like

$$T(t) = 76 + 23 \sin \left(\frac{2\pi}{365}(t - 108) \right)$$

- In one written report in which the students were required to explain their reasoning in creating each of the parameter values was the following passage: “*The frequency was the next value to determine. This was deceptively simple.*” A number of these college algebra students at a two year college subsequently presented their work at a series of annual conferences at a local university and one of these students was recruited to enter the graduate physics program at the university.
- Polynomial approximations to the sine and cosine based on trigonometric identities are introduced to reinforce both the identities and the behavior of the sinusoidal functions and polynomials, as well as setting the stage for the subsequent formal development of Taylor polynomials in calculus. One trigonometry unit problem on the examination was a routine question: “If the SIN and TAN buttons on your calculator are broken, find $\sin 40^\circ$ and $\tan 40^\circ$.” One student converted 40° to radians and estimated $\sin 40^\circ$ using a succession of polynomial approximations until he was convinced he had achieved an appropriate degree of convergence, and then found the tangent in the expected way. Other students have solved related problems on different exams at the college algebra and precalculus levels using comparable reasoning. If the hallmark of student learning is to develop the ability to apply what one has learned in novel ways in different situations, certainly these students have succeeded beyond the authors’ wildest imaginations.
- On the first test in Calculus I, after the students had seen families of functions and the concept of the derivative without any formal differentiation formulas, one of the authors presented the graph of the derivative of a function and asked where the unseen function achieves its maximum and minimum. The

16 What Does Conceptual Understanding Mean?

following reasoning was expected: The derivative is mostly positive, so the function is mostly increasing, and therefore it has its minimum at the left end-point and its maximum at the right. Of the twenty-eight students, nine gave the expected line of reasoning for a problem they had never seen. Fourteen independently used the graph of the derivative to sketch a graph of the actual function (reversing the process of graphical differentiation that they had seen); Thirteen of these students actually drew reasonably accurate sketches for f and used it appropriately. More significantly, under the pressure of an exam, these fourteen students created the concept of the antiderivative, a notion not previously mentioned. . .



The notion of approximating one function by another is a theme that makes an extraordinarily powerful impact on many students in calculus. In another Calculus I class, one of the authors stressed the idea of local linearity and pointed out that eventually the accuracy breaks down. Various students began suggesting ways to improve on the tangent line approximation, first by using circles, and then parabolas. At that point, the author introduced the idea of the Taylor polynomial approximation without the students seeing a single differentiation formula. This theme kept occurring repeatedly as the students interpreted almost everything during the entire year from this point of view. For example:

- While emphasizing the notion of local linearity early in calculus, one of the authors pointed out that no closed form solution exists for the many-body problem, where the spacecraft is subject to the gravity of the earth, the moon, and the sun. This was followed by a brief discussion of how NASA could use the idea of local linearity to calculate the orbits of spacecraft using Newton’s second law of motion by relating the acceleration to the sum of the forces, undoing the derivatives to get the position of the spacecraft, projecting along a short tangent line to the motion, and repeating the process to produce an essentially smooth curve. Another student (incidentally, the weakest in the group and the only one to withdraw from the course because she couldn’t keep up) asked: “*Couldn’t you improve on the accuracy by using a Taylor polynomial instead of the tangent line?*” Great idea! It’s known as the *Improved Euler Method* for the numerical solution of differential equations.
- Newton’s method was briefly introduced at the end of a class a month later, but there was no time for even a single example. One student stayed to ask

“Couldn’t you improve on the accuracy by using a Taylor polynomial instead of the tangent line?” Sure you can—that result is known as the *Euler Correction Formula*!

- Still later in the course, one of the authors mentioned that when you rotate the curve for $y = 1/x$ about the x -axis, the solid of revolution generated has finite volume, but infinite surface area. One student immediately tossed the following question out: *“Can you give me an example where the reverse is true—a solid with infinite volume, but with a finite surface area?”* Hmmm, that’s a good question! Based on the isoperimetric theorem from the calculus of variations, such a solid is impossible. However, that is a rather powerful theorem to have to bring to bear to answer a seemingly innocuous, but extremely unusual, question in freshman calculus! But, when one emphasizes conceptual understanding, the rules can change dramatically.

The above questions and comments are far from the only ones of such a penetrating and insightful nature. In fact, the authors have come to expect deeply thoughtful conceptual questions in all courses. These questions are far more telling than merely “seeing the answers” to questions on exams. This is well beyond learning mathematics; these students are creating mathematical concepts on the “fly,” often under the pressure of taking an exam. By traditional standards, many of these students would be deemed poor math students because their algebraic skills are relatively weak. They are certainly far less skilled than a TI-89 calculator, but they can think and create, things which that calculator cannot do!

All of these examples lead to a problem that has perplexed the present authors: How do we adequately reward a student for asking such a question or coming up with such a solution or making such a significant conceptual leap? We have given such students a bonus of 10 points, say, on an exam or a half-grade bump up in the final grade as a reward for such intellectual creativity. But it sometimes does not seem adequate; if a student shares the same mathematical insight of an Euler, a mere 10 extra points on a test seems rather inadequate. In one sense, we can argue that every one of these students has earned an automatic A for the course. Unfortunately, neither of us is comfortable with doing that, however right it sounds.

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References

- AMATYC (1995). *Crossroads in mathematics: Standards for introductory college mathematics before calculus*, Memphis, TN: AMATYC.
- Gordon, F. S. (2001). What does the slope mean? *PRIMUS*, *XI*, 258–264.
- Gordon, F. S. (2006). Assessing what students learn: Reform versus traditional precalculus and follow-up calculus. In N. B. Hastings, et al. (Eds). *A fresh start to collegiate mathematics: Rethinking the courses below calculus* (MAA Notes 69, pp. 181–192). Washington, DC: MAA.
- Mathematical Association of America (2004). *CUPM curriculum guide*. Washington, DC: Author.